

CONVEXITY PROPERTIES OF GRADIENT MAPS

PETER HEINZNER AND PATRICK SCHÜTZDELLER

ABSTRACT. We consider the action of a real reductive group G on a Kähler manifold Z which is the restriction of a holomorphic action of the complexified group $G^{\mathbb{C}}$. We assume that the induced action of a compatible maximal compact subgroup U of $G^{\mathbb{C}}$ on Z is Hamiltonian. We have an associated gradient map $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of \mathfrak{g} . For a G -stable subset Y of Z we consider convexity properties of the intersection of $\mu_{\mathfrak{p}}(Y)$ with a closed Weyl chamber in a maximal abelian subspace \mathfrak{a} of \mathfrak{p} . Our main result is a Convexity Theorem for real semi-algebraic subsets Y of $Z = \mathbb{P}(V)$ where V is a unitary representation of U .

1. INTRODUCTION

Let U be a compact Lie group and $U^{\mathbb{C}}$ its complexification. Then the map $\text{map } U \times \mathfrak{iu} \rightarrow U^{\mathbb{C}}, (u, \xi) \mapsto u \exp \xi$ is a diffeomorphism. A closed subgroup G of $U^{\mathbb{C}}$ with Lie algebra \mathfrak{g} is said to be compatible if the restriction $K \times \mathfrak{p} \rightarrow G$ is a diffeomorphism where $K = G \cap U$ and $\mathfrak{p} = \mathfrak{iu} \cap \mathfrak{g}$. In the rest of this paper we fix a compatible G a compact complex manifold Z and a holomorphic action $U^{\mathbb{C}} \times Z \rightarrow Z$. We also assume that there is a U -invariant Kähler form ω and a U -equivariant momentum map $\mu: Z \rightarrow \mathfrak{u}^*$. We fix a U -invariant inner product \langle, \rangle on $\mathfrak{u} \cong \mathfrak{iu}$ and view μ as a map from Z into \mathfrak{iu} . Since $\mathfrak{p} \subset \mathfrak{iu}$ the composition of μ with the orthogonal projection of \mathfrak{iu} onto \mathfrak{p} defines a K -equivariant map $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$ which we call the G -gradient map. Then we have $\text{grad } \mu_{\mathfrak{p}}^{\xi} = \xi_Z$ for $\xi \in \mathfrak{p}$ where grad is computed with respect to the Riemannian structure given by ω , $\mu_{\mathfrak{p}}^{\xi} := \langle \mu_{\mathfrak{p}}, \xi \rangle$ and ξ_Z is the vector field induced by the action. For a maximal dimensional Lie subalgebra \mathfrak{a} of \mathfrak{g} which is contained in \mathfrak{p} and a G -stable subset Y of Z we have the set $A(Y) := \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}$. In section 5 we prove the following

Theorem. *If Y is closed, then $A(Y)$ is a finite union of convex polytopes. Each of the polytopes is the convex hull of $\mu_{\mathfrak{p}}$ -images of A fixed points in Y where $A = \exp \mathfrak{a}$.*

Let \mathfrak{a}_+ be a positive Weyl chamber of \mathfrak{a} and set $A(Y)_+ := A(Y) \cap \mathfrak{a}_+$. The main result of this paper is the following

Both authors are partially supported by the Sonderforschungsbereich SFB TR 12 of the Deutsche Forschungsgemeinschaft.

Convexity Theorem. *Let $Z = \mathbb{P}(V)$ be the complex projective space of a unitary representation of U and ω the induced Kähler structure on $\mathbb{P}(V)$ with standard momentum map. Let Y be a closed real semi-algebraic subset of Z whose real algebraic Zariski closure is irreducible. Then $A(Y)_+$ is a convex polytope.*

Corollary. *Let $Z = U^\mathbb{C}/Q$ be a complex flag manifold endowed with a U -invariant Kähler structure and G a real form of $U^\mathbb{C}$. Then $A(\overline{G \cdot x})_+$ is a convex polytope for any $x \in Z$.*

All of the above are well known if G is a complex subgroup of $U^\mathbb{C}$ and Y is an irreducible complex analytic subset of Z . In this entirely holomorphic setup the Convexity Theorem holds for any compact Kähler manifold Z (see e.g. [GS05] for more on the history of the subject). On the other hand very little is known for a non complex group G and general Z . In this generality convexity of $A(Y)_+$ is known only in very special cases. See e.g. [Ko73] and [OS00].

2. BASIC PROPERTIES OF THE GRADIENT MAP

As before we assume that $U^\mathbb{C}$ acts holomorphically on a compact Kähler manifold Z and that the Kähler form ω is U -invariant. It is also assumed that there is a U -equivariant momentum map μ and we denote the associated G -gradient map by $\mu_{\mathfrak{p}}$. For the convenience of the reader we recall here several known basic facts which will be needed later.

For a subspace \mathfrak{m} of \mathfrak{g} and $z \in Z$ let $\mathfrak{m} \cdot z := \{\xi_Z(z) \mid \xi \in \mathfrak{m}\}$. The following elementary fact is shown in [HSch07].

Lemma 2.1. *We have $\ker d\mu_{\mathfrak{p}}(z) = (\mathfrak{p} \cdot z)^\perp$ for all $z \in Z$.*

Let $G = K \exp \mathfrak{p}$ be a compatible closed subgroup of $U^\mathbb{C}$. For $\beta \in \mathfrak{p}$ we set $\mathcal{M}_{\mathfrak{p}}(\beta) := \mu_{\mathfrak{p}}^{-1}(\beta) \subset Z$ and $\mathcal{M}_{\mathfrak{p}} := \mathcal{M}_{\mathfrak{p}}(0)$. For $z \in \mathcal{M}_{\mathfrak{p}}$ the isotropy group $G_z = K_z \exp \mathfrak{p}_z$ is a compatible subgroup of $U^\mathbb{C}$ ([HSch07, 5.5]). Since the G_z -representation on $T_z(Z)$ is completely reducible ([HSch07, 14.9]), there is a G_z -stable decomposition $T_z(Z) = \mathfrak{g} \cdot z \oplus W$. We have the following general Slice Theorem ([HSch07, 14.10, 14.21]):

Theorem 2.2 (Slice Theorem). *Let $z \in \mathcal{M}_{\mathfrak{p}}$. Then there exists a G_z -stable open neighborhood S of $0 \in W$, a G -stable open neighborhood Ω of $z \in Z$ and a G -equivariant diffeomorphism $\Psi: G \times^{G_z} S \rightarrow \Omega$ where $\Psi([e, 0]) = z$ and $G \times^{G_z} S$ denotes the G -bundle associated with the principal bundle $G \rightarrow G/G_z$.*

Actually, we have a Slice Theorem at every $z \in Z$. Set $\beta := \mu_{\mathfrak{p}}(z)$ and let $G^\beta = \{g \in G : \text{Ad } g \cdot \beta = \beta\}$ denote the centralizer of β . Then we have a slice for the action of G^β , as follows.

The centralizer G^β is a compatible subgroup of $U^\mathbb{C}$ with Cartan decomposition $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ where $K^\beta = K \cap G^\beta$ and $\mathfrak{p}^\beta = \{\xi \in \mathfrak{p} : \text{ad}(\xi)\beta = 0\}$.

The group G^β is also compatible with the Cartan decomposition of the centralizer $(U^\mathbb{C})^\beta = (U^\beta)^\mathbb{C}$ and β is fixed by the action of U^β on \mathfrak{u}^β . This implies that the \mathfrak{u}^β -component of μ defines a U^β -equivariant shifted momentum map $\widehat{\mu_{\mathfrak{u}^\beta}}: Z \rightarrow \mathfrak{u}^\beta$, $\widehat{\mu_{\mathfrak{u}^\beta}}(z) = \mu_{\mathfrak{u}^\beta}(z) - \beta$. The associated G^β -gradient map is given by $\widehat{\mu_{\mathfrak{p}^\beta}}: Z \rightarrow \mathfrak{p}^\beta$, $\widehat{\mu_{\mathfrak{p}^\beta}}(z) = \mu_{\mathfrak{p}^\beta}(z) - \beta$. This shows that the Slice Theorem applies to the action of G^β at every point $z \in (\widehat{\mu_{\mathfrak{p}^\beta}})^{-1}(0) = \mathcal{M}_{\mathfrak{p}^\beta}(\beta)$. In particular, if G is commutative, then we have a Slice Theorem for G at every point of Z .

3. ORBIT-TYPE STRATIFICATION

Let Y be a closed G -stable subset of Z . Our initial goal is to show that $A(Y) = \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}$ is a finite union of convex polytopes. The proof is completed in section 5. It depends on the orbit type stratification of Z with respect to compatible commutative subgroups of G which we explain now.

Let \mathfrak{b} be a Lie subalgebra of \mathfrak{g} which is contained in \mathfrak{p} . Note that $B = \exp \mathfrak{b}$ is compatible with the Cartan decomposition of $U^\mathbb{C}$ and that $\exp: \mathfrak{b} \rightarrow B$ is an isomorphism of commutative Lie groups. Let Z^B denote the set of B -fixed points in Z . Since $B = \exp \mathfrak{b}$ we have $Z^B = Z^{\mathfrak{b}} = \{x \in Z : \xi_Z(x) = 0 \text{ for all } \xi \in \mathfrak{b}\}$. Further let $\mu_{\mathfrak{b}}: Z \rightarrow \mathfrak{b}$ the composition of μ with the orthogonal projection of \mathfrak{u} onto \mathfrak{b} .

Lemma 3.1. *For $B = \exp \mathfrak{b}$ we have*

- (1) *the set Z^B of B -fixed points is a smooth complex submanifold of Z and*
- (2) *the B -gradient map $\mu_{\mathfrak{b}}|_{Z^B}: Z^B \rightarrow \mathfrak{b}$ is locally constant.*

Proof. Since B acts on Z by holomorphic transformations the set Z^B is a complex subspace of Z . The isotropy representation defines a linear B -action on $T_x(Z)$. By the Slice Theorem a B -stable open neighborhood of x is B -equivariantly diffeomorphic to an open neighborhood of 0 in $T_x(Z)$. Since the set of fixed points of a linear action is a linear subspace the set Z^B is smooth. This shows (1). The second assertion follows from Lemma 2.1. \square

For a connected subgroup B of $A = \exp(\mathfrak{a})$ let $Z^{(B)} := \{z \in Z : A_z = B\}$. The group B is compatible and we have $Z^{(B)} = Z^{(\mathfrak{b})} := \{z \in Z : \mathfrak{a}_z = \mathfrak{b}\}$. A connected component S of $Z^{(B)}$ is called an A -stratum of type A/B or alternatively an \mathfrak{a} -stratum of type $\mathfrak{a}/\mathfrak{b}$.

Lemma 3.2. *Let S be an \mathfrak{a} -stratum of type $\mathfrak{a}/\mathfrak{b}$, $q \in \mu_{\mathfrak{a}}(S)$ and $\mathfrak{a}(S) := q + \mathfrak{b}^\perp$. Then we have:*

- (1) *S is open in Z^B .*
- (2) *$\mu_{\mathfrak{a}}(S)$ is an open subset of $\mathfrak{a}(S)$.*
- (3) *$\mu_{\mathfrak{a}}: S \rightarrow \mathfrak{a}(S)$ is a submersion.*

Proof. Any $x \in Z^{(B)}$ has an open A -stable neighborhood Ω which is A -equivariantly diffeomorphic to an A -stable neighborhood of $[e, 0]$ in $A \times^B W$

where W is a B -representation space and e is the neutral element in A (Slice Theorem). The A -stratum in $A \times^B W$ of type A/B is given by $A \times^B W^B$ and coincides with the set of B -fixed points in $A \times^B W$. This shows that $Z^{(B)}$ is open in Z^B . In particular S is open in Z^B and we have (1).

With respect to the orthogonal decomposition $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{b}^\perp$ we have $q = q_{\mathfrak{b}} + q_{\mathfrak{b}^\perp} \in \mu_{\mathfrak{a}}(S)$ and $\mu_{\mathfrak{a}} = \mu_{\mathfrak{b}} \oplus \mu_{\mathfrak{b}^\perp}$. We may also replace Z by Z^B and $U^{\mathbb{C}}$ by the analytic Zariski closure of A in $U^{\mathbb{C}}$ without changing our assumptions. With this in mind we have $\ker d\mu_{\mathfrak{a}}(x) = (\mathfrak{a} \cdot x)^\perp$ for all $x \in Z = Z^B$. This implies that $d\mu_{\mathfrak{b}^\perp}(x): T_x(Z^{(B)}) \rightarrow \mathfrak{b}^\perp$ is a bijection for all $x \in Z^{(B)}$. Since $\mu_{\mathfrak{b}}: Z^B \rightarrow \mathfrak{b}$ is locally constant on Z^B this shows (3) and also (2). \square

Let A^c be the analytic Zariski closure of A in $U^{\mathbb{C}}$ and \mathfrak{a}^c its Lie algebra. The group A^c is a complex reductive compatible subgroup of $U^{\mathbb{C}}$ with maximal compact subgroup $T = A^c \cap U$. We have $A^c = T \exp(\mathfrak{t})$ where \mathfrak{t} denotes the Lie algebra of T . In the following \overline{S} denotes the topological closure of a subset S of Z in Z . The same notation is used for a subset of \mathfrak{a} or more generally for subsets of a given topological space.

Lemma 3.3. *Let S be an \mathfrak{a} -stratum of type $\mathfrak{a}/\mathfrak{b}$ in Z . Then*

- (1) *S is an A^c -stable locally closed complex submanifold of Z*
- (2) *$\mu_{\mathfrak{a}}(\overline{S}) = \overline{\mu_{\mathfrak{a}}(S)}$ is a convex polytope.*
- (3) *every y in \overline{S} which is mapped by $\mu_{\mathfrak{a}}$ onto a vertex of $\mu_{\mathfrak{a}}(\overline{S})$ is an A -fixed point.*

Proof. Since A^c is connected and $A_{g \cdot y} = (A^c)_{g \cdot y} \cap A = (A^c)_y \cap A = A_y$ holds for all $g \in A^c$ and $y \in Z$ we have (1).

Note that any A -stratum S is an A^c -stable Kählerian submanifold of Z . Let $\mu_{\mathfrak{t}}: Z \rightarrow \mathfrak{t}^*$ denote the momentum map on Z given by restricting $\mu: Z \rightarrow \mathfrak{u}^*$ to \mathfrak{t} . In [HH96] it is shown that $\overline{\mu_{\mathfrak{t}}(S)}$ is a convex polytope in \mathfrak{t}^* . Equivalently $\overline{\mu_{\mathfrak{it}}(S)} = \mu_{\mathfrak{it}}(\overline{S})$ is a convex polytope, where $\mu_{\mathfrak{it}}: Z \rightarrow \mathfrak{t}^*$ is the A^c -gradient map given by μ . Since $\mu_{\mathfrak{a}}$ is the composition of $\mu_{\mathfrak{it}}$ and the orthogonal projection of it onto \mathfrak{a} this shows that $\mu_{\mathfrak{a}}(\overline{S})$ is a convex polytope in \mathfrak{a} . Finally it is shown in [HSSt07] that every $y \in S$ whose image is a vertex of $\mu_{\mathfrak{a}}(\overline{S})$ has to be an A -fixed point. \square

For the following Lemma we recall that we assume the G -action and therefore also the A -action on Z to be effective.

Lemma 3.4. (1) *There are only finitely many A -strata.*

- (2) *The A -stable subset of Z where A acts freely is the unique open A -stratum is given by $S_0 = \{z \in Z : \mathfrak{a}_z = \{0\}\}$ and is open and dense in Z .*
- (3) *Z is the disjoint union of A -strata.*
- (4) *The boundary $\overline{S} \setminus S$ of an A -stratum is a finite union of A -strata \tilde{S} such that $\dim \tilde{S} < \dim S$ holds.*

Proof. This follows from compactness of Z and the Slice Theorem. \square

Lemma 3.5. *Let $S \neq S_0$ be an \mathfrak{a} -stratum of type $\mathfrak{a}/\mathfrak{b}$ and $y \in S$. Then there are \mathfrak{a} -strata S_j of type $\mathfrak{a}/\mathfrak{a}_j$, $j = 1, \dots, r$ such that $y \in \overline{S_j}$, $\dim \mathfrak{a}_j = 1$ and $\mathfrak{b} = \mathfrak{a}_1 + \dots + \mathfrak{a}_r$ hold.*

Proof. We fix a point $y \in S$ and apply the Slice Theorem to the A -action on Z at y . This means that we find an A -stable open neighborhood Ω of y , a $B := A_y$ subrepresentation W of the isotropy representation $T_y(Z)$, an open B -stable neighborhood Ω_W of $0 \in W$ and an A -equivariant diffeomorphism $\Psi: A \times^B \Omega_W \rightarrow \Omega$ such that $\Psi([e, 0]) = y$. Since the A -action on Z is assumed to be effective and since it is real analytic the B -action on W is effective. We view $A \times^B \Omega_W$ as an open subset of $A \times^B W$ and note that a \mathfrak{b} -stratum $S(W)$ in W of type $\mathfrak{b}/\mathfrak{c}$ determines uniquely the \mathfrak{a} -stratum $A \times^B S(W) \subset A \times^B W$ of type $\mathfrak{a}/\mathfrak{c}$. This implies that we may restrict our attention to the \mathfrak{b} representation W .

The image of B in $\mathrm{GL}(W)$ is real diagonalizable since B acts on $T_x(Z)$ by selfadjoint operators ([HSch07]). Let $W = W_{\chi_0} \oplus \dots \oplus W_{\chi_r}$ be the isotypical decomposition of W where for any linear function $\chi: \mathfrak{b} \rightarrow \mathbb{R}$ we set $W_\chi = \{w \in W : \xi \cdot w = \chi(\xi)w \text{ for all } \xi \in \mathfrak{b}\}$ and χ_0 denotes the zero map. We have $W_{\chi_0} = W^{\mathfrak{b}}$. The open \mathfrak{b} -stratum in W is of type \mathfrak{b} and contains $W^{\mathfrak{b}} \times (W_{\chi_1} \setminus \{0\}) \times \dots \times (W_{\chi_r} \setminus \{0\})$. Then

- a) Any \mathfrak{b} -stratum has 0 in its closure and
- b) if 0 does not lie in the open \mathfrak{b} -stratum, then there are \mathfrak{b} -strata $S_j(W)$ of type $\mathfrak{b}/\mathfrak{c}_j$, $j = 1, \dots, l$ such that $\dim \mathfrak{c}_j = 1$ and $\mathfrak{b} = \mathfrak{c}_1 \oplus \dots \oplus \mathfrak{c}_l$.

This follows from the fact that the B -representation W is diagonalizable. Since the B -action on W is effective the open \mathfrak{b} -stratum of W is of type \mathfrak{b} . \square

4. DECOMPOSITION OF THE GRADIENT MAP IMAGE

As in the previous section let \mathfrak{a} be a linear subspace of \mathfrak{p} which is a subalgebra of \mathfrak{g} and $A = \exp \mathfrak{a}$ the corresponding commutative compatible subgroup of G . Let S be an \mathfrak{a} -stratum of type $\mathfrak{a}/\mathfrak{b}$. We set $\sigma := \mu_{\mathfrak{a}}(S)$ and let $\mathfrak{a}(S) := \mathfrak{a}(\sigma)$ be the unique affine subspace of \mathfrak{a} which contains σ as an open subset (Lemma 3.3). We have $\mathfrak{a}(S) = \mathfrak{a}(\sigma) = q + \mathfrak{b}^\perp$ for any $q \in \mu_{\mathfrak{a}}(\overline{S})$ where $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{b}^\perp$. Since \mathfrak{b} only depends on $\mathfrak{a}(\sigma)$ we will also use the notation $\mathfrak{a}_\sigma = \mathfrak{b}$. Formulated in more geometric terms \mathfrak{a}_σ is the linear subspaces of \mathfrak{a} which is perpendicular to the affine linear space $\mathfrak{a}(\sigma)$ and coincides with the isotropy Lie algebra of any point $z \in S$.

Let $\Sigma := \{\mathfrak{a}(\sigma) : S \text{ is an } A\text{-stratum and } \sigma = \mu_{\mathfrak{a}}(S)\}$ denote the set of all affine subspaces of \mathfrak{a} obtained in this way. For the open A -stratum S_0 we have $\mathfrak{a} = \mathfrak{a}(\sigma_0)$ and σ_0 is the interior of $P := \mu_{\mathfrak{a}}(Z)$. Let $\Sigma_1 := \{\sigma \in \Sigma : \mathrm{codim}_{\mathfrak{a}} \mathfrak{a}(\sigma) = 1\}$ and $P_0 := P \setminus \bigcup_{\sigma \in \Sigma_1} P \cap \mathfrak{a}(\sigma)$.

Lemma 4.1. *The set P_0 is open in \mathfrak{a} .*

Proof. It is sufficient to show that every face F of $P = \mu_{\mathfrak{a}}(Z)$ of codimension one is contained in $\mathfrak{a}(\sigma)$ for some $\sigma \in \Sigma_1$.

The image $\mu_{\mathfrak{a}}(S) = \sigma$ of any \mathfrak{a} -stratum is open in $\mathfrak{a}(\sigma)$. If we apply this to the open stratum S_0 we see that for any face $F \neq P$ of P this implies that $S_0 \cap \mu_{\mathfrak{a}}^{-1}(F) = \emptyset$. Since we have only finitely many \mathfrak{a} -strata this shows for a face F with $\text{codim}_{\mathfrak{a}} F = 1$ that there is an \mathfrak{a} -stratum S_F with $\sigma_F \in \Sigma_1$ and that σ_F is open in F . We have $F \subset \mathfrak{a}(\sigma_F)$ and therefore $P \setminus \bigcup_{\sigma_F} \mathfrak{a}(\sigma_F)$ is open in \mathfrak{a} where the union is over all faces of P which are of codimension one. This implies that P_0 is open in \mathfrak{a} . \square

As in the previous section let S_0 denote the unique open \mathfrak{a} -stratum in Z .

Lemma 4.2. *We have $\mu_{\mathfrak{a}}^{-1}(P_0) \subset S_0$ or equivalently $\mathfrak{a}_y = \{0\}$ for all $y \in \mu_{\mathfrak{a}}^{-1}(P_0)$.*

Proof. Assume that there is a $y \in \mu_{\mathfrak{a}}^{-1}(P_0)$ such that $\mathfrak{a}_y \neq 0$. Let \tilde{S} be the \mathfrak{a} -stratum which contains y . Since \tilde{S} is not the open \mathfrak{a} -stratum there is an \mathfrak{a} -stratum S of type $\mathfrak{a}/\mathfrak{a}_1$ where $\dim \mathfrak{a}_1 = \dim \mathfrak{a} - 1$ such that $\tilde{S} \subset \overline{S}$. This shows that $\mu_{\mathfrak{a}}(y) \in \overline{\sigma} \subset \mathfrak{a}(\sigma)$ for $\sigma = \mu_{\mathfrak{a}}(S)$. Since $\sigma \in \Sigma_1$ this contradicts the definition of P_0 . \square

Let $C(P_0)$ denote the set of connected components of P_0 . For $\gamma \in C(P_0)$ let $P(\gamma)$ be the closure of the connected component γ . The set $P(\gamma)$ is a convex polytope with non-empty interior $\text{int}_{\mathfrak{a}}(P(\gamma))$ in \mathfrak{a} . Let $\mathcal{F}(P_0) := \{F : F \text{ is a face of } P(\gamma) \text{ where } \gamma \in C(P_0)\}$ be the set of faces which are determined by P_0 . We have $P_0 = \bigcup_{\gamma \in C(P_0)} \text{int}_{\mathfrak{a}}(P(\gamma))$ and $P = \bigcup_{\gamma \in C(P_0)} P(\gamma)$. More importantly every face $F \in \mathcal{F}(P_0)$ of codimension one is given by $P(\gamma) \cap \mathfrak{a}(\sigma)$ for some $\sigma \in \Sigma_1$ and $\gamma \in C(P_0)$.

For a convex polytope F in \mathfrak{a} we introduce the following notation. The affine span of F is denoted by $\mathfrak{a}(F)$ and $\text{int}(F) = \text{int}_{\mathfrak{a}(F)}(F)$ denotes the interior of F as a subspace of $\mathfrak{a}(F)$. The linear subspace of \mathfrak{a} which is perpendicular to $\mathfrak{a}(F)$ is denoted by \mathfrak{a}_F . The dimension of F is denoted by $\dim F$ as is by definition the dimension of $\mathfrak{a}(F)$. Similarly $\text{codim } F$ means the codimension of $\mathfrak{a}(F)$ as a subspace of \mathfrak{a} and coincides with the dimension of \mathfrak{a}_F .

For $\gamma \in C(P_0)$ let $\Sigma_1(\gamma)$ denote the set of codimension one faces of $P(\gamma)$.

Proposition 4.3. *Let $\gamma \in C(P_0)$ and let F be a face of $P(\gamma)$ of codimension k . Then there are $\sigma_1, \dots, \sigma_k \in \Sigma_1(\gamma)$ such that*

- (1) $F = P(\gamma) \cap \mathfrak{a}(\sigma_1) \cap \dots \cap \mathfrak{a}(\sigma_k)$ and
- (2) $\mathfrak{a}_y \subset \mathfrak{a}_F = \mathfrak{a}_{\sigma_1} + \dots + \mathfrak{a}_{\sigma_k}$ for all $q \in \text{int}(F)$ and $y \in \mu_{\mathfrak{a}}^{-1}(q)$.

Proof. Property (1) follows from the definition of $P(\gamma)$.

Let $q \in \text{int}(F)$ and $y \in \mu_{\mathfrak{a}}^{-1}(q)$. Since $\sigma_1, \dots, \sigma_k \in \Sigma_1(\gamma)$ we have $\mathfrak{a}(F) = \mathfrak{a}(\sigma_1) \cap \dots \cap \mathfrak{a}(\sigma_k)$. We have to show that $\mathfrak{a}_y \subset \mathfrak{a}_{\sigma_1} + \dots + \mathfrak{a}_{\sigma_k}$. Since $q \in \text{int}(F)$ we have $\sum_{\sigma \in \Sigma_1, q \in \mathfrak{a}(\sigma)} \mathfrak{a}_{\sigma} = \mathfrak{a}_{\sigma_1} + \dots + \mathfrak{a}_{\sigma_k}$. The Slice Theorem implies that $\mathfrak{a}_y = \sum_{\tilde{\sigma} \in \tilde{\Sigma}} \mathfrak{a}_{\tilde{\sigma}}$ for some subset $\tilde{\Sigma} \subset \Sigma_1$ (Lemma 3.5). This gives $\mathfrak{a}_y \subset \mathfrak{a}_{\sigma_1} + \dots + \mathfrak{a}_{\sigma_k}$. \square

By the construction of the polytopes $P(\gamma)$ the set $\mathcal{F}(P_0)$ is closed under intersection and we have the following

Remark 4.4. *Let D be the finite union of elements in $\mathcal{F}(P_0)$. Then the set of all points $\xi \in D$ such that D is non convex in any neighborhood of ξ is again a finite union of elements in $\mathcal{F}(P_0)$.*

5. SEMISTABLE POINTS AND CONVEXITY

In this section we show that convexity of $A(Y)_+$ is closely related to the behavior of semistable points after shifting.

Let β be a point in \mathfrak{p} . The U -orbit $U \cdot \beta \subset \mathfrak{iu}$ can be identified with the coadjoint orbit $U \cdot \mathfrak{i}\beta \subset \mathfrak{u}$ and is a complex flag manifold $O := U^\mathbb{C}/Q$, where $Q := \{g \in U^\mathbb{C} \mid \lim_{t \rightarrow -\infty} \exp(t\beta) \cdot g \cdot \exp(-t\beta) \text{ exists in } U^\mathbb{C}\}$. In particular, this induces a Kähler structure and a holomorphic $U^\mathbb{C}$ -action on $U \cdot \beta$. We denote this action by $(g, x) \mapsto g \bullet x$. The G -gradient map on $U^\mathbb{C} \bullet \beta$ is then just given by the projection of $O = U \cdot \beta \subset \mathfrak{iu}$ onto \mathfrak{p} .

Proposition 5.1. ([HSt05]) *For $\beta \in \mathfrak{p}$ we have $G \bullet \beta = K \cdot \beta$ in O .*

The G -gradient map $\mu_{\mathfrak{p},\beta} : Z \times U^\mathbb{C} \bullet \beta \rightarrow \mathfrak{p}, (z, \xi) \mapsto \mu_{\mathfrak{p}}(z) - \pi_{\mathfrak{p}}(\xi)$ is called the shifting of $\mu_{\mathfrak{p}}$ with respect to β where $\pi_{\mathfrak{p}} : \mathfrak{iu} \rightarrow \mathfrak{p}$ denotes the orthogonal projection. In particular, β is contained in the image of $\mu_{\mathfrak{p}} : Z \rightarrow \mathfrak{p}$ if and only if 0 is contained in the image of $\mu_{\mathfrak{p},\beta} : Z \times U^\mathbb{C} \bullet \beta \rightarrow \mathfrak{p}$. The set of semistable points in $Y \times G \bullet \beta \subset Y \times U^\mathbb{C} \bullet \beta$ with respect to the value α is by definition the set

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha})(Y \times G \bullet \beta) := \{(y, \xi) \in Y \times G \bullet \beta \mid \overline{G \bullet (y, \xi)} \cap (\mu_{\mathfrak{p},\beta})^{-1}(\alpha) \neq \emptyset\}$$

for any $\alpha \in \mathfrak{a}$. For $\alpha = 0$ we set $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},0})(Y \times G \bullet \beta) = \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})(Y \times G \bullet \beta)$. With this notation we have the following.

Theorem 5.2. *Let Y be a closed G -stable subset of Z such that the intersection*

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_1})(Y \times G \bullet \beta) \cap \mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_2})(Y \times G \bullet \beta)$$

is nonempty for any $\alpha_j \in A_+(Y)$ and $\beta \in \mathfrak{a}$ with $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_j})(Y \times G \bullet \beta) \neq \emptyset$. Then $A_+(Y)$ is a convex polytope.

For the proof of the theorem we need some preparation.

Lemma 5.3. *Let \mathfrak{a}_+ be a closed Weyl-chamber in \mathfrak{a} and $q, p \in \mathfrak{a}_+$. Then*

$$\|k \cdot q - p\|^2 \geq \|q - p\|^2$$

holds for all $k \in K$.

Proof. Since the inner product on \mathfrak{p} is K -invariant we have

$$\|k \cdot q - p\|^2 - \|q - p\|^2 = -2 \cdot \langle k \cdot q - q, p \rangle.$$

We have $\langle k \cdot q, p \rangle = \langle \pi_{\mathfrak{a}}(k \cdot q), p \rangle$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection of \mathfrak{p} onto \mathfrak{a} . But $\pi_{\mathfrak{a}}(k \cdot q)$ is contained in the convex hull of the orbit of the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ through q ([Ko73]) and therefore it suffices

to prove the inequality $\langle w \cdot q - q, p \rangle \leq 0$ for all $w \in W$. Let $\alpha \in \Sigma^+$ and σ_α be the corresponding simple reflection. Then we have $\langle q, \alpha \rangle \geq 0$ and therefore $\langle q, -\alpha \rangle = \langle q, \sigma_\alpha \cdot \alpha \rangle = \langle \sigma_\alpha \cdot q, \alpha \rangle \leq 0$ which implies $\sigma_\alpha \cdot q - q = -\lambda\alpha$ for some positive number λ . Since every Weyl group element can be written as a product of these simple reflections $w \cdot q - q$ is a negative linear combination of positive roots which shows $\langle w \cdot q - q, p \rangle \leq 0$ for all $w \in W$. \square

Remark 5.4. (1) *If one reads through our paper in the case that $G = A$, then one obtains a proof of the result we needed from Kostant's paper [Ko73]. Thus our results are independent of [Ko73].*
 (2) *Kostant's fundamental paper was the first paper containing convexity results in the spirit presented here.*

Proposition 5.5. *Let $p_0 \in \mathfrak{a}$ and assume that $q_0 \in A(Y) := \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}$ is a minimum of the function $\psi_{p_0}: A(Y) \rightarrow \mathbb{R}, q \mapsto \|q - p_0\|$. Then*

$$(\mu_{\mathfrak{p}}|_Y)^{-1}(q_0) \subset Y^\xi := \{y \in Y \mid \xi_Z(y) = 0\}$$

for $\xi := q_0 - p_0$.

Proof. We claim that q_0 is also a minimum of the function $\tilde{\psi}_{p_0}: \mu_{\mathfrak{p}}(Y) \rightarrow \mathbb{R}, q \mapsto \|q - p_0\|$. Let \mathfrak{a}_+ be a Weyl chamber such that $p_0 \in \mathfrak{a}_+$. Since $\mu_{\mathfrak{p}}(Y)$ is K -stable we have $\mu_{\mathfrak{p}}(Y) = K \cdot A(Y) = K \cdot (A(Y) \cap \mathfrak{a}_+)$. Let $\tilde{q} \in \mu_{\mathfrak{p}}(Y)$ and $k \in K$ such that $\tilde{q} = k \cdot q$ where $q \in \mathfrak{a}_+$. This implies $\|\tilde{q} - p_0\| \geq \|q - p_0\| \geq \|q_0 - p_0\|$ (Lemma 5.3).

Let $y \in (\mu_{\mathfrak{p}}|_Y)^{-1}(q_0)$ and $\xi = q_0 - p_0$. Then y is a critical point of the function $\eta: G \cdot y \rightarrow \mathbb{R}, g \cdot y \mapsto \frac{1}{2}\|\mu_{\mathfrak{p}}(g \cdot y) - q_0\|^2$. Now $0 = d\eta(y) = \langle d\mu_{\mathfrak{p}}(y), \mu_{\mathfrak{p}}(y) - p_0 \rangle = d\mu^\xi(y)$ implies $\xi_Z(y) = 0$. \square

Proposition 5.6. *We have $A(Y) \cap F = F$ for all faces $F \in \mathcal{F}(P_0)$ such that $\text{int}(F) \cap A(Y) \neq \emptyset$.*

We will prove the proposition recursively by arguing by dimension of the faces of $\mathcal{F}(P_0)$ and starting with those faces which are of maximal dimension. In order to carry this out we note the following

Lemma 5.7. *Let $F^* \in \mathcal{F}(P_0)$. Then $F^* \cap A(Y) = F^*$ implies that $F \cap A(Y) = F$ for all $F \in \mathcal{F}(P_0)$ which are contained in F^* .* \square

Proof of Proposition 5.6. Let F be an arbitrary face such that $\text{int}_{\mathfrak{a}(F)}(F) \cap A(Y) \neq \emptyset$. By the above indicated induction we may assume that our Proposition holds for all faces $F^* \in \mathcal{F}(P_0)$ with $\dim F^* > \dim F$. Lemma 5.7 implies that we additionally may assume that $\text{int}(F^*) \cap A(Y) = \emptyset$ for all faces F^* which properly contain our given face F . The advantage of this assumption is that for any $q_1 \in \text{int}(F) \cap A(Y)$ we can find a $r > 0$ such that $A(Y) \cap \Delta_r(q_1) = \text{int}(F) \cap A(Y) \cap \Delta_r(q_1)$ holds. For any $p_1 \in \Delta_{\frac{r}{2}}(q_1)$ such that $\xi_1 := p_1 - q_1 \in \mathfrak{a}_F$ is perpendicular to our face F we obtain $\|p_1 - q_1\| \leq \|p_1 - q\|$ for all $q \in \Delta_r(q_1) \cap A(Y)$. Proposition 5.5 shows that

$(\xi_1)_Z(y) = 0$ for all such ξ_1 and $y \in \mu_{\mathfrak{a}}^{-1}(q_1)$. Since $q_1 \in \text{int}(F) \cap A(Y)$ was arbitrary this shows $\mathfrak{a}_F \subset \mathfrak{a}_y$ for all $y \in \mu_{\mathfrak{a}}^{-1}(\text{int}(F) \cap A(Y))$. Now Proposition 4.3 implies $\mathfrak{a}_y = \mathfrak{a}_F$ for all $y \in \mu_{\mathfrak{a}}^{-1}(\text{int}(F) \cap A(Y))$.

We will now argue that $\mathfrak{a}_y = \mathfrak{a}_F$ together with the assumption that $\text{int}(F) \cap A(Y) \neq \emptyset$ and $F \cap A(Y) \neq F$ leads to a contradiction.

Assume that there is a $q_1 \in \text{bd}_{\text{int}(F)}(\text{int}(F) \cap A(Y)) := (\text{int}(F) \cap A(Y)) \setminus \text{int}_{\mathfrak{a}(F)}(F \cap A(Y))$ and let $r > 0$ such that $\Delta_r(q_1) \cap F \subset \text{int}(F)$ and $\Delta_r(q_1) \cap A(Y) = F \cap \Delta_r(q_1) \cap A(Y)$ hold. Here we use the assumption that $\text{int}(F^*) \cap A(Y) = \emptyset$ for all faces F^* which properly contain F . Then there is a $p_0 \in \Delta_{\frac{r}{2}}(q_1) \cap F$ with $p_0 \notin A(Y)$ and therefore a $q_0 \in A(Y) \cap \Delta_r(q_1)$ with satisfies $\|p_0 - q_0\| \leq \|p_0 - q\|$ for all $q \in A(Y) \cap \text{int}(F) \cap \Delta_r(q_1)$. Since $A(Y) \cap \text{int}(F) \cap \Delta_r(q_1) = A(Y) \cap \Delta_r(q_1)$ we may apply Proposition 5.5. This gives $\xi_Z(y) = 0$ where $\xi = p_0 - q_0$ and $y \in \mu_{\mathfrak{a}}^{-1}(q_0)$. Since $\xi \neq 0$ and $\xi \notin \mathfrak{a}_Y$ this contradicts $\mathfrak{a}_F = \mathfrak{a}_y$. \square

Corollary 5.8. *The set $A(Y)$ is a union of faces $F \in \mathcal{F}(P_0)$ and is therefore a finite union of convex polytopes each of it the convex hull of images of fixed points of T in Y .* \square

For the proof of Theorem 5.2 we also need the the fact that a subset D of an Euclidian vector space which is a finite union of convex polytopes and is not convex has the property that for any sufficiently small $r > 0$ there exists a point $\beta \in \mathfrak{a}$ such that the closed ball of radius r and center β meets D in precisely two points α_1 and α_2 . This geometric input has also been used in Kirwan's proof of her convexity result ([Kir84b]).

Proof of Theorem 5.2. The set $A_+(Y)$ is a finite union of convex polytopes (Corollary 5.8). Assume that $A_+(Y)$ is not convex. Then there exist $r > 0$ and $\beta \in \mathfrak{a}$ such that the closed ball of radius r and center β meets $A_+(Y)$ in precisely two points α_1 and α_2 which are on the boundary of this ball. Now if for $\alpha \in \mathfrak{a}$ the value $\|\alpha\|$ is critical for the function $\|\mu_{\mathfrak{p},\beta}\|^2: Z \times G \bullet \beta \rightarrow \mathbb{R}$ then there is an associated pre-stratum $S_\alpha = \{w \in Z \times G \bullet \beta : \|\alpha\| = \min\{\|\mu_{\mathfrak{p},\beta}(g \cdot w)\| : g \in G\}\}$ for the G -action on $Z \times G \bullet \beta$ in the sense of [HSS07]. The values $\|\alpha_j\|$ are critical points of $\|\mu_{\mathfrak{p},\beta}\|^2: Z \times G \bullet \beta \rightarrow \mathbb{R}$ and define two non empty G -pre-strata for the G -action on $Z \times G \bullet \beta$. We have $S_{\alpha_j} \cap Y = \mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_j})(Y \times G \bullet \beta)$. Since $\alpha_1 \neq \alpha_2$ and $\alpha_j \in \mathfrak{a}_+$ these pre-strata are disjoint. Consequently we have

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_1})(Y \times G \bullet \beta) \cap \mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_2})(Y \times G \bullet \beta) = \emptyset.$$

This contradicts the assumption of Theorem 5.2. \square

Remark 5.9. *In the case where $G = U^\mathbb{C}$ and Y is an irreducible $U^\mathbb{C}$ -stable complex subspace of Z the set $Y \times U^\mathbb{C} \bullet \beta$ is a Kählerian space and the pre-strata in the above proof are locally closed complex subspaces of $Y \times U^\mathbb{C} \bullet \beta$. This implies that there is a unique open $U^\mathbb{C}$ -stratum which is dense in $Z \times U^\mathbb{C} \bullet \beta$. The above proof then gives Kirwan's convexity theorem for actions of complex reductive groups on compact $U^\mathbb{C}$ -stable irreducible*

complex subspaces of Kähler manifolds. This is a rather special case of the more general convexity result in [HH96].

6. THE PROJECTIVE CASE

We fix now a finite dimensional unitary representation space V of the compact group U and consider $Z = \mathbb{P}(V)$. The action of U on V extends to a holomorphic linear action of $U^\mathbb{C}$ and induces an algebraic $U^\mathbb{C}$ -action on the associated complex projective space $\mathbb{P}(V)$. There are G -gradient maps $\mu_{\mathfrak{p},V}: V \rightarrow \mathfrak{p}$, $\mu_{\mathfrak{p},V}^\xi(v) = \langle \xi v, v \rangle$ on V with respect to the Kähler structure induced by the unitary one on V and a G -gradient map $\mu_{\mathfrak{p},\mathbb{P}(V)}: \mathbb{P}(V) \rightarrow \mathfrak{p}$, $\mu_{\mathfrak{p},\mathbb{P}(V)}^\xi([v]) = \frac{\langle \xi v, v \rangle}{\|v\|^2}$ on $\mathbb{P}(V)$ with respect to the induced Fubini-Study Kählerian structure on $\mathbb{P}(V)$. Here we denote the fixed positive Hermitian structure on V by $\langle \cdot, \cdot \rangle$ and $[v] \in \mathbb{P}(V)$ denotes the line through $v \in V \setminus \{0\}$. Note that the Fubini-Study form on $\mathbb{P}(V)$ is given by symplectic reduction and is up to a positive constant the unique Kähler form on $\mathbb{P}(V)$ which is invariant with respect to the natural action of the special unitary group $SU(V)$.

In order to simplicity the notation we set $\mu_{\mathfrak{p}} := \mu_{\mathfrak{p},\mathbb{P}(V)}$. We view $\mathbb{P}(V)$ as a real algebraic variety and fix a G -stable closed real semialgebraic subset Y of $\mathbb{P}(V)$. We say that Y is irreducible if the real Zariski closure of Y in $\mathbb{P}(V)$ is a real irreducible subvariety. Our main result is

Theorem 6.1. *The set $A_+(Y) := \mu_{\mathfrak{p}}(Y) \cap \mathfrak{a}_+$ is a convex polytope.*

We have the following consequences which are shown below.

Corollary 6.2. *Let $Z = U^\mathbb{C}/Q$ be a complex flag manifold with G -gradient map $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$. Then the sets $A_+(Z)$ and $A_+(\overline{G \cdot x})$ are convex polytopes.*

Using this fact we also have

Corollary 6.3. *Let $Z = U^\mathbb{C}/Q$ be a complex flag manifold and assume that G is a real form of $U^\mathbb{C}$ which is given as the set of fixed points of an anti-holomorphic involution commuting with the given Cartan involution on $U^\mathbb{C}$. Let $\xi \in A_+(Z)$ be the unique closest point to the origin. The set of semistable points $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\xi})(Z) := \{z \in Z \mid \overline{G \cdot z} \cap \mu_{\mathfrak{p}}^{-1}(\xi) \neq \emptyset\}$ coincides with the union of all open G -orbits in Z . Moreover, the closed $K^\mathbb{C}$ -orbits have the same image under the G -gradient map $\mu_{\mathfrak{p}}$.*

To prove Theorem 6.1 we use the same strategy as in the proof of Theorem 5.2. For this we need the notion of quasi-rational points in \mathfrak{a} . Let \mathfrak{h} be a maximal abelian subalgebra of the centralizer $\mathfrak{z}_{\mathfrak{t}}(\mathfrak{a})$. Then $\mathfrak{s}_{\mathfrak{u}} := \mathfrak{h} \oplus \mathfrak{ia}$ is maximal torus in \mathfrak{u} . We call a point $\alpha \in \mathfrak{a} \simeq \mathfrak{ia}$ quasi-integral if α is the projection of an integral element α' in the compact torus $\mathfrak{s}_{\mathfrak{u}} = \mathfrak{h} \oplus \mathfrak{ia}$ onto \mathfrak{ia} . We denote this projection by $\pi_{\mathfrak{ia}}$. A point $\beta \in \mathfrak{ia}$ is called quasi-rational if it is a rational multiple of an quasi-integral element $\alpha \in \mathfrak{a}$. The following lemma allows us to make a reduction to quasi-rational points.

Lemma 6.4. *The set $A(Y)_+$ is a finite union of quasi-rational polytopes, i. e. the polytopes are convex hulls of finitely many quasi-rational points. In particular, the quasi-rational points are dense in $A(Y)_+$.*

Proof. Corollary 5.8 says that $A(Y)_+$ is the intersection of a positive Weyl chamber with a finite union of convex polytopes which are given by the convex hull of images of sets of A -fixed points in Y . If $[v] \in Y^A$. Then v is contained in a weight space $V_\chi := \{v \in V \mid \xi \cdot v = \chi(\xi) \cdot v \ \forall \xi \in \mathfrak{a}\}$ of the \mathfrak{a} -representation V . Here $\chi: \mathfrak{a} \rightarrow \mathbb{R}$ is a linear function with $\chi = i\varphi_*|_{\mathfrak{ia}}$ for some character $\varphi: S_U \rightarrow S^1$. Here S_U denotes the maximal torus of U with Lie algebra \mathfrak{s}_u and S^1 is the maximal compact subgroup of \mathbb{C}^* . For every $\xi \in \mathfrak{a}$ we therefore have $\mu_{\mathfrak{a}}([v])(\xi) = \chi(\xi)$. So $\mu_{\mathfrak{a}}([v])$ is an quasi-integral element in \mathfrak{a} in the sense of the appendix. Consequently $A(Y)_+$ is a finite union of quasi-rational convex polytopes. \square

Any semialgebraic set Y has a finite semialgebraic stratification, i. e. Y can be decomposed into real analytic locally closed semialgebraic submanifolds A_i such that for $\overline{A_i} \cap A_j \neq \emptyset$ we have $A_j \subset \overline{A_i}$ and $\dim A_j < \dim A_i$. We choose one decomposition, define \hat{Y} to be the union of the maximal dimensional strata and set $\dim Y := \dim \hat{Y}$. If Y is a real algebraic set this gives the Krull dimension of Y . In general we have $\dim Y = \dim cl(Y)$ where $cl(Y)$ denotes the real Zariski closure of Y . For detailed proofs see e. g. [BR] or [C00].

Let Y be a closed G -stable irreducible semialgebraic subset of $\mathbb{P}(V)$. The technical part of the proof of Theorem 6.1 is the following

Proposition 6.5. *Let $\alpha \in \mathfrak{a}_+$ and $\beta \in \mathfrak{a}$ such that*

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha})(Y \times G \bullet \beta)$$

is non empty. Further assume that β is quasi-rational. Then

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha})(Y \times G \bullet \beta) \cap (\hat{Y} \times G \bullet \beta)$$

is open and dense in $\hat{Y} \times G \bullet \beta$.

Actually the assumption that β is quasi-rational is not necessary but simplifies the proof.

Proof of Theorem 6.1. Assume $A_+(Y)$ is non convex. As in the proof of Theorem 5.2 we get points $\alpha_1, \alpha_2 \in \mathfrak{a}_+$ and $\beta \in \mathfrak{a}$ such that

$$\mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_1})(Y \times G \bullet \beta) \cap \mathcal{S}_G(\mathcal{M}_{\mathfrak{p},\alpha_2})(Y \times G \bullet \beta) = \emptyset.$$

Using Lemma 6.4 and Remark 4.4 one can choose the point β to be quasi-rational. See [Kir84b] for the explicit construction. But this contradicts Proposition 6.5 and shows the assertion. \square

Proof of Proposition 6.5. Since β is quasi-rational, there exists an integral element $\delta' \in \mathfrak{s}_u$ such that $\beta = \frac{1}{n} \cdot \pi_{\mathfrak{ia}}(\delta')$ for some $n \in \mathbb{N}$. Therefore we

have a G -equivariant diffeomorphism $\varphi: Y \times G \bullet \beta' \rightarrow Y' \times G \bullet \delta'$ given by multiplication with n in the second component. Here $\beta' = \frac{1}{n} \cdot \delta'$ and $Y' = Y$ but seen as a subset of $\mathbb{P}(V)$ endowed with the Fubini Study form multiplied with n . In particular, we have

$$\mathcal{M}_{\mathbf{p},\alpha}(Y \times G \bullet \beta) = \mathcal{M}_{\mathbf{p},\alpha}(Y \times G \bullet \beta') = \varphi^{-1}(\mathcal{M}_{\mathbf{p},n \cdot \alpha}(Y' \times G \bullet \delta)).$$

Since φ is G -equivariant, we get the analog equation for the sets of semistable points. Since $Y' \times G \bullet \delta'$ is a closed G -stable irreducible semialgebraic set of some projective space it suffices to prove the proposition for $\beta = 0$.

So let $\alpha \in \mathfrak{a}_+$ such that $\mathcal{M}_{\mathbf{p},\alpha}(Y)$ is non empty. Then there exists a point $y \in Y$ such that $\mu_{\mathbf{p}}(y) = \alpha$. Since the quasi-rational points are dense in $A_+(Y)$ there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $\mu_{\mathbf{p}}(y_n) =: \alpha_n$ are quasi-rational and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$. In particular $\mathcal{M}_{\mathbf{p},\alpha_n}(Y)$ is non empty.

Let assume that the open set $\mathcal{S}_G(\mathcal{M}_{\mathbf{p},\alpha})(Y) \cap \hat{Y}$ is not dense in \hat{Y} . Then there exists a G -stable open subset U in the complement and an $r > 0$ such the $\|\mu_{\mathbf{p}}(y') - \alpha\|^2 \geq r$ for all $y' \in U$. Moreover, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|\mu_{\mathbf{p}}(y') - \alpha_n\|^2 \geq \frac{r}{2}$ for all y' in the G -stable subset U . Therefore, for $n \geq N$, the set $\mathcal{S}_G(\mathcal{M}_{\mathbf{p},\alpha_n})(Y) \cap \hat{Y}$ is a non empty open subset of \hat{Y} which is not dense. Therefore it suffices to prove the assertion for quasi-rational α .

If $\alpha \in \mathfrak{a}_+$ is a quasi-rational point, i. e. $\alpha = \frac{p}{q} \cdot \gamma \in \mathfrak{a}_+$ for some quasi-integral element γ and some coprime integers p and q , it follows from the results given in the second part of the appendix that $Y_\alpha = \Phi_\alpha(Y \times K \cdot [v'_\gamma])$ is a closed G -stable irreducible semialgebraic subset of $\mathbb{P}(V_\alpha)$. Here γ' again is a integral element in \mathfrak{s}_u over γ . Note that the projection of $\Phi_\alpha^{-1}(\mathcal{S}_G(\mathcal{M}_{\mathbf{p}})(Y_\alpha) \cap \hat{Y}_\alpha) \subset Y \times [v_\gamma]$ is just $\mathcal{S}_G(\mathcal{M}_{\mathbf{p}}(\alpha))(Y) \cap \hat{Y}$ where $\hat{Y}_\alpha = \Phi_\alpha(\hat{Y} \times K \cdot [v_\gamma])$. Therefore we can also restrict to the case $\alpha = 0$.

We have $\overline{\mathcal{M}_{\mathbf{p}}(Y)} \neq \emptyset$ and $\mathcal{S}_G(\mathcal{M}_{\mathbf{p}})(Y) = Y \setminus (Y \cap \pi(\mathcal{N}_G))$ where $\mathcal{N}_G := \{v \in V \mid 0 \in \overline{G \cdot v}\}$ is the null cone in V . Since the null cone is a real algebraic subset of V (Lemma 7.1) and $cl(Y)$ is irreducible, the intersection $\pi(\mathcal{N}_G) \cap cl(Y)$ is either $cl(Y)$ or a proper algebraic subset of lower dimension in $cl(Y)$. Since $\dim \hat{Y} = \dim Y = \dim cl(Y)$ and $\mathcal{M}_{\mathbf{p}}(Y) \neq \emptyset$ the set $\hat{Y} \cap \pi(\mathcal{N}_G)$ is a proper semialgebraic subset of lower dimension in \hat{Y} . In particular, its complement $\mathcal{S}_G(\mathcal{M}_{\mathbf{p}}(Y)) \cap \hat{Y}$ is open and dense in \hat{Y} (see e. g. [BR] for basic properties of semialgebraic sets).

□

We now prove the two corollaries 6.2 and 6.3 about complex flag manifolds. Note first that every complex flag manifold $Z = U^\mathbb{C}/Q$ can be identified with an orbit $U \cdot \beta$ where β is contained in the cone $C_Q := \{\lambda \in \mathfrak{s}_u \mid \lambda = \sum c_j \cdot \alpha_j, \alpha_j \in \Pi', c_j \in \mathbb{R}^+\}$ where Π' is the subset of the simple roots which define Q . Using the notation of the second part of the appendix we have the following

Corollary 6.6. *Let $Z = U^\mathbb{C}/Q$ be a complex flag manifold and let Y be a closed G -stable subset such that $\varphi_\beta(Y)$ is an irreducible semialgebraic subset of $\mathbb{P}(\Gamma_\beta)$ for every integral β in the cone C_Q . Then $A_+(Y)$ is a convex polytope for every G -gradient map $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$ on Z .*

Proof. Every momentum map on the complex flag manifold Z with respect to the U -action is of the form $\mu: Z \rightarrow \mathfrak{u}$, $x \mapsto \alpha_x$, where α is an element in the cone C_Q . By Theorem 6.1, the lemma holds if α is an integral element in C_Q . If α is a rational point, it can be written in the form $\alpha = \frac{1}{n} \cdot \beta$ for some integral element β in $\mathfrak{s}_{\mathfrak{u}}$ and some $n \in \mathbb{N}$. In particular, $\mu_{\mathfrak{p}}(Y) = \frac{1}{n} \cdot \tilde{\mu}_{\mathfrak{p}}(Y)$, where $\tilde{\mu}$ is given by $\tilde{\mu}(x) = \beta_x$. This proves the rational case. Since the rational points are dense in each cone C_Q , we can construct a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of rational points $\alpha_n \in \mathfrak{s}_{\mathfrak{u}}$ such that $Q_-(\alpha_n)$ coincides with Q for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$. This gives a sequence of momentum maps $(\mu^{(n)})_{n \in \mathbb{N}}$ such that the assertion holds for every element of this sequence. This gives the corollary for the limit point μ . \square

In particular, the above corollary can be applied to the complex flag manifold Z itself and closures of G -orbits in Z which gives Corollary 6.2.

Proof of Corollary 6.3. Since ξ is the unique closest point to the origin in $A_+(Z)$, a point $x_0 \in (\mu_{\mathfrak{p}})^{-1}(\xi)$ is a global minimum of $\eta_{\mathfrak{p}}$. Therefore, the G -orbit through x_0 is open and the $K^\mathbb{C}$ -orbit through x_0 is closed and coincides with the K -orbit through x_0 (see [BL02] and [MUV92]). So we need to show that all open G -orbits are contained in the set $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\xi))$. By Proposition 6.5, we know that $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\xi))$ is open and dense in Z in the integral case. This can be extended to the non integral case as in the proof of Proposition 6.5. Consequently, the set $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\xi))$ contains all open G -orbits and the closed $K^\mathbb{C}$ -orbits in these open G -orbits are contained in $(\mu_{\mathfrak{p}})^{-1}(K \cdot \xi)$. \square

7. APPENDIX

7.1. Algebraicity of the null cone. In the proof of Proposition 6.5 we use the algebraicity of the null cone $\mathcal{N}_G = \{v \in V \mid 0 \in \overline{G \cdot v}\}$. Here we give a proof of this fact.

Lemma 7.1. *The null cone \mathcal{N}_G is an algebraic subset of V .*

Proof. Using the decomposition of G into its semisimple part G_s and its center $Z(G)$ (see [HSt05]) we first prove the algebraicity of the null cone \mathcal{N}_{G_s} with respect to G_s by using results of [B] and [RS]. All of them appear as special cases in [HSch07].

Let $G_s^\mathbb{C}$ be the complexification of G_s and let $V^\mathbb{C}$ the corresponding complexified representation space. Further let F_1, \dots, F_d be the generators of the algebra $\mathbb{C}[V^\mathbb{C}]^{G_s^\mathbb{C}}$ of $G_s^\mathbb{C}$ -invariant polynomials on $V^\mathbb{C}$. Then the map $F = (F_1, \dots, F_d): V^\mathbb{C} \rightarrow \mathbb{C}^d$, parameterizes the Zariski closed $G_s^\mathbb{C}$ -orbits in

$V^{\mathbb{C}}$. Without loosing generality we can assume that $F_j(0) = 0$ for all generators F_j .

By a result of [B], for every $v \in V$ the orbit $G_s^{\mathbb{C}} \cdot v$ is Zariski closed in $V^{\mathbb{C}}$ if $G_s \cdot v$ is closed in V . Therefore, for every closed orbit $G_s \cdot v$, $v \neq 0$, there exists a function F_j from the list of generators of the algebra $\mathbb{C}[V^{\mathbb{C}}]^{G_s^{\mathbb{C}}}$ such that $0 \neq F_j|_{G_s \cdot v}$. Since G_s is Zariski dense in $G_s^{\mathbb{C}}$, we can assume that the polynomials F_j are extensions of real polynomials f_1, \dots, f_d which generate the algebra $\mathbb{R}[V]^{G_s}$. In particular, we have $0 \neq f_j|_{G_s \cdot v}$ which shows that the null cone is given as the real algebraic subset $\{v \in V \mid f_j(v) = 0, j = 1, \dots, d\}$ of V .

To prove the general case let V_j denote the weight spaces of $Z(G)$ in V . Since G_s and $Z(G)$ commute, these subspaces are stable under G_s . In particular, we have $\mathbb{R}[V]^{G_s} = \bigotimes \mathbb{R}[V_j]^{G_s}$. Each factor $\mathbb{R}[V_j]^{G_s}$ has finitely many generators which can be chosen to be homogeneous polynomials. Let $f: V \rightarrow \mathbb{R}^k$ be the polynomial map which is given by all these generators. Then f is invariant with respect to G_s and equivariant with respect to $Z(G)$. Here the action of $Z(G)$ on \mathbb{R}^k is given by the action on each generator. The corresponding null cone of $Z(G)$ in \mathbb{R}^k is a finite union of linear subspaces $H_j \subset \mathbb{R}^k$ (see [HSch07] Corollary 15.5). Therefore, the preimage of this null cone under f is an algebraic subset of V which we call \mathcal{N}' . We show that the null cone \mathcal{N}_G coincides with this algebraic set \mathcal{N}' .

Let G_U denote the maximal compact subgroup of $G_s^{\mathbb{C}}$ and let H be a G_U -invariant positive definite Hermitian form on $V^{\mathbb{C}}$ such that the alternating part vanishes on V . Such a form exists by [RS] and we get a momentum map $\mu_{G_U}: V^{\mathbb{C}} \rightarrow \mathfrak{g}_U^*$ for the G_U action on $V^{\mathbb{C}}$. By construction, $\tilde{\mathcal{M}}_{\mathfrak{p}} = \mathcal{M} \cap V$ where $\mathcal{M} := (\mu_{G_U})^{-1}(0)$.

We have $\mathcal{N}_G \subset \mathcal{N}'$. For the opposite inclusion let $v \in \mathcal{N}'$. By definition, there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset Z(G)$ such that $\lim_{n \rightarrow \infty} g_n \cdot f(v) = 0$. For every $g_n \cdot f(v) \in \mathbb{R}^k$ let α_n be a point in a closed G_s -orbit in $f^{-1}(g_n \cdot f(v))$. By [RS] every closed G_s -orbit intersects the set $\tilde{\mathcal{M}}_{\mathfrak{p}}$ and we may choose $\alpha_n \in \tilde{\mathcal{M}}_{\mathfrak{p}}$. Let F be the complex extension of f to $V^{\mathbb{C}}$. The map f is a proper map when restricted to \mathcal{M} (see [RS]). This is also true for the restriction of F to $\tilde{\mathcal{M}}_{\mathfrak{p}}$ since $\tilde{\mathcal{M}}_{\mathfrak{p}} = \mathcal{M} \cap V$. Consequently the sequence $(\alpha_n)_{n \in \mathbb{N}}$ has a convergent subsequence with limit point $\alpha \in \tilde{\mathcal{M}}_{\mathfrak{p}}$. But $f(\alpha) = 0$ which implies $\alpha \in \mathcal{N}_{G_s}$ and consequently $v \in \mathcal{N}_G$. \square

7.2. Shifting with respect to quasi-rational points. Since there is in general no symplectic embedding of the orbit $U \cdot \beta \subset \text{iu}$ into a projective space $\mathbb{P}(W)$, equipped with the Fubini-Study metric coming from a U -invariant Hermitian form on W , the set $Y \times G \bullet \beta$ is in general not contained in the class of examples we are considering in section 6 of this paper. Therefore, we have to introduce a slight modification of the shifting procedure.

Given an integral element $\alpha' \in \mathfrak{s}_{\mathfrak{u}} \subset \mathfrak{s}$ we get an associated character $\chi_{\alpha'}: Q \rightarrow \mathbb{C}^*$ on the parabolic subgroup and a $U^{\mathbb{C}}$ -homogeneous line bundle

$$L^{\alpha} = U^{\mathbb{C}} \times_{\chi_{\alpha'}} \mathbb{C} = (U^{\mathbb{C}} \times \mathbb{C})/Q$$

over $U^{\mathbb{C}}/Q$. Let $\Gamma_{\alpha'} := \Gamma(U^{\mathbb{C}}/Q, L^{\alpha'})$ denote the space of holomorphic sections of the line bundle $L^{\alpha'}$ and let $\Gamma_{\alpha'}^*$ denote its dual space. By the theorem of Borel and Weil, the space $\Gamma_{\alpha'}$ is an irreducible $U^{\mathbb{C}}$ -representation space with highest weight α' and it follows that the projective space $\mathbb{P}(\Gamma_{\alpha'}^*)$ contains a unique complex U -orbit $U \cdot [v_{\alpha'}]$ with $U \cdot [v_{\alpha'}] \simeq U^{\mathbb{C}}/Q$. In particular this gives an embedding of $U \cdot \alpha'$ into a projective space. For detailed proofs see e.g. [Akh95] or [Huc01].

Now, let $\beta \in \mathfrak{a}$ be a quasi-rational point and let α be the unique minimal quasi-integral element in $\mathbb{R}^+ \cdot \beta$. Then $\beta = \frac{p}{q} \cdot \alpha$ for coprime natural numbers p and q . Let α' be an integral element in $\mathfrak{s}_{\mathfrak{u}}$ with $\pi_{\text{id}}(\alpha') = \alpha$. The momentum map $\mu_{\mathbb{P}(\Gamma_{-\alpha'}^*)}$ restricted to the orbit $U \cdot [v_{-\alpha'}]$ is given by $\mu_{\mathbb{P}(\Gamma_{-\alpha'}^*)}([v]) = \mu_{\mathbb{P}(\Gamma_{-\alpha'}^*)}(u \cdot [v_{\alpha'}]) = -u \cdot \alpha' =: -\alpha'_{[v]}$. Since β and α are related by the fixed numbers p and q , we can define a unique embedding

$$\Phi_{\beta}: Y \times U \cdot \alpha' \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(\Gamma_{-\alpha'}^*) \hookrightarrow \mathbb{P}(V^{\otimes q} \otimes (\Gamma_{-\alpha'}^*)^{\otimes p}) =: \mathbb{P}(V_{\beta})$$

using the Segre embedding. Define $Y_{\beta} := \Phi_{\beta}(Y \times G \cdot \alpha')$. Since $G \cdot \alpha'$ is a real algebraic set we may choose \hat{Y}_{β} to be $\Phi_{\beta}(\hat{Y} \times G \cdot \alpha')$. We get a K -equivariant map

$$\mu_{\mathfrak{p}, \beta}: Y_{\beta} \rightarrow \mathfrak{p}^*, \quad \Phi_{\beta}((y, \xi)) \mapsto q \cdot \mu_{\mathfrak{p}}(y) - p \cdot \xi$$

which we call the shifting of $\mu_{\mathfrak{p}}$ with respect to the quasi-rational point $\beta \in \mathfrak{a}$. In particular, this is contained in the class of examples which we consider in section 6.

REFERENCES

- [Akh95] D. N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, E27, Friedr. Vieweg & Sohn, Braunschweig, 1995.
- [Ati82] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.
- [BR] R. Benedetti and J.-J. Risler, *Real algebraic and semi-algebraic sets*, Actualités Mathématiques, Hermann, Paris, 1990.
- [B] D. Birkes, *Orbits of linear algebraic groups*, Ann. of Math. (2) **93** (1971), 459–475.
- [BL02] R. Bremigan and J. Lorch, *Orbit duality for flag manifolds*, Manuscripta Math. **109** (2002), no. 2, 233–261.
- [Ch46] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, Princeton, 1946.
- [C00] M. Coste, *An introduction to semialgebraic geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.
- [GS05] V. Guillemin and R. Sjamaar, *Convexity properties of Hamiltonian group actions*, CRM Monograph Series, vol. 26, American Mathematical Society, Providence, RI, 2005.

- [GS82] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.
- [HH96] P. Heinzner and A. Huckleberry, *Kählerian potentials and convexity properties of the moment map*, Invent. Math. **126** (1996), no. 1, 65–84.
- [HSch07] P. Heinzner and G. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), no. 1, 197–232.
- [HSt07] P. Heinzner, G. Schwarz and H. Stötzel, *Stratifications with respect to actions of real reductive groups*, arXiv:math.CV/0611491.
- [HSt05] P. Heinzner and H. Stötzel, *Semistable points with respect to real forms*, Math. Ann., DOI: 10.1007/s00208-006-0063-1.
- [Ho65] G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.
- [Huc01] A. T. Huckleberry, *Introduction to group actions in symplectic and complex geometry*, Infinite dimensional Kähler manifolds (Oberwolfach, 1995), DMV Sem., vol. 31, Birkhäuser, Basel, 2001, pp. 1–129.
- [Ki84] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, NJ, 1984.
- [Kir84b] ———, *Convexity properties of the moment mapping. III*, Invent. Math. **77** (1984), no. 3, 547–552.
- [Kn96] A. W. Knap, *Lie Groups Beyond an Introduction*, Progress in Mathematics **140**, Birkhäuser, Boston, 1996.
- [Ko73] B. Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 413–455.
- [Ma01] A. Marian, *On the real moment map*, Math. Res. Lett. **8** (2001), 779–788.
- [Ma82] T. Matsuki, *Orbits on affine symmetric spaces under the action of parabolic subgroups*, Hiroshima Math. J. **12** (1982), 307–320.
- [OS00] L. O’Shea and R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. **317** (2000), no. 3, 415–457.
- [RS] R. W. Richardson and P. J. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. (2) **42** (1990), no. 3, 409–429.
- [MUV92] I. Mirković, T. Uzawa and K. Vilonen, *Matsuki correspondence for sheaves*, Invent. Math. **109** (1992), 231–245.
- [W] J. A. Wolf, *The action of a real semisimple group on a complex flag manifold. I. Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

FAKULTÄT UND INSTITUT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM

E-mail address: heinzner@cplx.rub.de

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT PADERBORN, D-33095 PADERBORN

E-mail address: schuetzd@math.upb.de